On the unification of classical and novel integrable surfaces: I. Differential geometry

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Abstract

A novel class of integrable surfaces is recorded. This class of O surfaces is shown to include and generalize classical surfaces such as isothermic, constant mean curvature, minimal, 'linear' Weingarten, Guichard and Petot surfaces and surfaces of constant Gaußian curvature. It is demonstrated that the construction of a Bäcklund transformation for O surfaces leads in a natural manner to an associated parameter-dependent linear representation. The classical pseudosphere and breather pseudospherical surfaces are generated.

1 Introduction

It was Jonas (1915) who pointed out that the Bäcklund transformations and associated permutability theorems which had been established for a variety of classes of surfaces around the turn of the century have a common origin. Amongst these transformations are those by Ribaucour and Koenigs which preserve lines of curvature and conjugate nets governed by Laplace equations with equal point invariants respectively. Both transformations constitute particular cases of a transformation which was termed 'Fundamental transformation' by Eisenhart (1962) in his treatise 'Transformations of Surfaces'. The Fundamental transformation and its cousins have since then been recognized as central in the geometric and algebraic construction of Bäcklund and Darboux-type transformations and their applications in soliton theory (Matveev & Salle 1991; Rogers & Shadwick 1982; Rogers & Schief 2000). In this connection, it is observed that the celebrated coherent structure solutions (dromions) of the Davey-Stewartson I equation (Boiti *et al.* 1988) have been derived by a Darboux-type transformation which represents nothing but a variant of the Fundamental transformation.

In an attempt to complement Jonas' fundamental contribution, we here embark on a study of the common origin of classes of surfaces which are invariant under the Fundamental transformation. Thus, we consider sets of n surfaces in

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a Euclidean space $\mathbb{R}^3$ which are related by the classical Combescure transformation defining parallel conjugate nets. These may be canonically associated with three Combescure-related surfaces in a dual (pseudo-)Euclidean space $\mathbb{R}^n$. We then isolate a privileged class of surfaces (O surfaces) by demanding that the surfaces in both $\mathbb{R}^3$ and $\mathbb{R}^n$ be parametrized in terms of orthogonal coordinates. Remarkably, it turns out that this class of O surfaces encapsulates as canonical reductions classical surfaces such as isothermic, constant mean curvature, minimal, 'linear' Weingarten, Guichard and Petot surfaces and surfaces of constant Gaußian curvature. These are obtained by specifying appropriately the dimension and the metric of the dual space $\mathbb{R}^n$.

It is no accident that Guichard surfaces arise in this context. Thus, in a classical note in *Comptes Rendus de l'Académie des Sciences*, Guichard (1900) characterizes these surfaces in the following manner:

"Il existe une surface (N') *ayant même image sphérique de ses lignes de courbure que la surface* (N) *et telle que si* R_1 *et* R_2 *sont les rayons de courbure principaux de* (N), R_1' *et* R_2' *les rayons correspondants de* (N'), *on ait*

$$\mathrm{R}_1\mathrm{R}_2' + \mathrm{R}_2\mathrm{R}_1' = \mathrm{const.},$$

la constante n'étant pas nulle."

It will be demonstrated that the latter condition may be interpreted as a particular orthogonality constraint associated with O surfaces. In the light of this interpretation, Guichard's characterization may be regarded as containing the essence underlying the definition of O surfaces.

A Bäcklund transformation for O surfaces is obtained by constraining the Fundamental transformation in such a way that the above-mentioned orthogonality conditions are preserved. As a by-product, a matrix Lax pair for O surfaces is derived. As an application of the Bäcklund transformation for O surfaces, the classical pseudosphere and breather pseudospherical surfaces are generated. The Gauß-Mainardi-Codazzi equations for O surfaces are set down and it is shown that these may be regarded as the compatibility condition for a parameter-dependent linear system which generalizes the classical linear representation for isothermic surfaces set down by Darboux (1899) (Eisenhart 1962).

It is important to note that the formalism developed in this paper may readily be adapted to the case of integrable difference geometry (Bobenko & Seiler 1999). This may be regarded as a first step towards a unified description of integrability-preserving discretizations of differential geometries. In particular, integrable difference-geometric analogues of the above-mentioned classical surfaces are constructed without difficulty. This is the subject of a forthcoming article (Schief 2000b).

2 Conjugate coordinates and the Combescure transformation

In the following, we are concerned with the geometry of surfaces in a three-dimensional Euclidean space. If a surface $\Sigma \subset \mathbb{R}^3$ is parametrized in terms of (local) coordinates (x, y) in such a way that the position vector $\vec{R} = \vec{R}(x, y)$ to the surface Σ obeys a linear hyperbolic equation of the form

$$\vec{R}_{xy} = a\vec{R}_x + b\vec{R}_y \tag{1}$$

then the curves $x = \text{const}$ and $y = \text{const}$ are said to form a *conjugate net* (Eisenhart 1962) on Σ. In this case, it is convenient to introduce the parametrization

$$a = (\ln H)_y, \quad b = (\ln K)_x \tag{2}$$

and tangent vectors $\vec{X}, \vec{Y}$ according to

$$\vec{R}_x = \vec{X}H, \quad \vec{R}_y = \vec{Y}K. \tag{3}$$

The second-order equation (1) may then be brought into the first-order form

$$\vec{X}_y = q\vec{Y}, \quad \vec{Y}_x = p\vec{X}, \tag{4}$$

where the coefficients p and q are defined by

$$H_y = pK, \quad K_x = qH. \tag{5}$$

The latter system may be regarded as *adjoint* to the linear system (4).

Conversely, if $\{\vec{X}, \vec{Y}, H, K\}$ constitutes a solution of the linear system (4), (5) for some functions p and q then the relations (3) are compatible and $\vec{R}$ may be interpreted as the position vector of a surface $\Sigma \subset \mathbb{R}^3$ parametrized in terms of conjugate coordinates. A second solution $\{H_*, K_*\}$ of the adjoint system (5) gives rise to a second surface Σ_*, the position vector of which is defined by

$$\vec{R}_{*x} = \vec{X}H_*, \quad \vec{R}_{*y} = \vec{Y}K_*. \tag{6}$$

Accordingly, at corresponding points, the tangent vectors to the coordinate lines on the surfaces Σ and Σ_* are parallel. The surface Σ_* is termed a *Combescure transform* (Eisenhart 1962) of the surface Σ. Hence, the Combescure transformation maps conjugate nets to *parallel*[2] conjugate nets. It is important to note that the unit normal $\vec{N}$ to Σ and its Combescure transform $\vec{N}_*$ coincide, that is $\vec{N}_* = \vec{N}$.

It is evident that *lines of curvature* (Eisenhart 1960), which are uniquely defined by the requirement that they be conjugate and orthogonal, are also preserved by the Combescure transformation. A particular Combescure transform $\Sigma_\circ$ of a surface Σ parametrized in terms of curvature coordinates is therefore

[2]This notion of parallelism is not to be confused with the definition of parallel surfaces.

given by its spherical representation, that is the parametrized sphere swept out by the unit normal $\vec{N}$ to Σ. Indeed, the linear system (4) implies that

$$\vec{N}_x \cdot \vec{Y} = 0, \quad \vec{N}_y \cdot \vec{X} = 0 \tag{7}$$

and hence there exist functions $H_\circ$ and $K_\circ$ such that

$$\vec{N}_x = \vec{X} H_\circ, \quad \vec{N}_y = \vec{Y} K_\circ \tag{8}$$

by virtue of the orthogonality condition $\vec{X} \cdot \vec{Y} = 0$. Thus, the coordinate system on $\Sigma_\circ$ generated by the position vector $\vec{R}_\circ = \vec{N}$ is conjugate and parallel to that on Σ. In fact, the relations (8) constitute the well-known Rodrigues formulae (Eisenhart 1960) if one expresses $H_\circ$ and $K_\circ$ in terms of the principal curvatures (cf. §4).

3 Combescure-related surfaces and their duals

It is natural to investigate the properties of sets $\{\Sigma_1, \ldots, \Sigma_n\}$ of surfaces which are related by Combescure transformations. To this end, we consider the linear systems

$$\begin{aligned} \vec{X}_y &= q\vec{Y}, \quad & \underline{H}_y &= p\underline{K} \\ \vec{Y}_x &= p\vec{X}, \quad & \underline{K}_x &= q\underline{H}, \end{aligned} \tag{9}$$

where $\vec{X}, \vec{Y} \in \mathbb{R}^3$ and $\underline{H}, \underline{K} \in \mathbb{R}^n$ are interpreted as column and row vectors respectively, and define a matrix $\underline{\vec{R}} \in \mathbb{R}^{3,n}$ via the compatible equations

$$\underline{\vec{R}}_x = \vec{X}\underline{H}, \quad \underline{\vec{R}}_y = \vec{Y}\underline{K}. \tag{10}$$

Thus, the geometric interpretation given below is immediate:

The vectors

$$\vec{R}_\kappa \in \mathbb{R}^3, \quad \kappa = 1, \ldots, n$$

parametrize parallel conjugate nets on surfaces $\Sigma_\kappa \subset \mathbb{R}^3$ *with tangent vectors* $\vec{X}$ *and* $\vec{Y}$.

However, since there exists complete symmetry between $\{\vec{X}, \vec{Y}\}$ and $\{\underline{H}, \underline{K}\}$ and the definition of conjugate nets is in fact independent of the dimension of the ambient space, the following point of view is also valid:

The vectors

$$\underline{R}^k \in \mathbb{R}^n, \quad k = 1, 2, 3$$

parametrize parallel conjugate nets on surfaces $\Sigma^k \subset \mathbb{R}^n$ *with tangent vectors* $\underline{H}$ *and* $\underline{K}$.

We refer to the surfaces Σ^k as *dual* to the surfaces Σ_κ. As mentioned earlier, we here regard the ambient space $\mathbb{R}^3$ as a Euclidean space even though the generalization to pseudo-Euclidean spaces $\mathbb{R}^3$ and their higher-dimensional

analogues is straightforward. By contrast, it turns out pivotal to deal with pseudo-Euclidean dual spaces $\mathbb{R}^n$. Thus, we endow $\mathbb{R}^n$ with the inner product

$$\underline{H} \cdot \underline{K} = \underline{H}\,\underline{K}^{\intercal} = \sum_{\kappa,\mu=1}^{n} H_\kappa S^{\kappa\mu} K_\mu, \tag{11}$$

where $S = (S^{\kappa\mu})$ is a constant symmetric matrix. Moreover, it is noted that the coordinate lines and the associated tangent vectors on the surfaces Σ_κ may be reparametrized according to

$$\begin{aligned}(\partial_x, \vec{X}, q) &\to f(x)(\partial_x, \vec{X}, q)\\ (\partial_y, \vec{Y}, p) &\to g(y)(\partial_y, \vec{Y}, p)\end{aligned} \tag{12}$$

without changing the tangent vectors $\underline{H}$ and $\underline{K}$ on the dual surfaces Σ^k. Analogously, the change of variables

$$\begin{aligned}(\partial_x, \underline{H}, p) &\to \tilde{f}(x)(\partial_x, \underline{H}, p)\\ (\partial_y, \underline{K}, q) &\to \tilde{g}(y)(\partial_y, \underline{K}, q)\end{aligned} \tag{13}$$

preserves the tangent vectors $\vec{X}$ and $\vec{Y}$.

The concept of dual conjugate nets is implicit in the work of Darboux (1910). It has been exploited in the context of integrable differential/difference geometries by several authors (Konopelchenko & Schief 1998; Doliwa & Santini 1999). The significance of matrix 'bilinear potentials' such as $\underline{\vec{R}}$ in connection with the iteration of the classical Fundamental transformation and its relatives has also been discussed (Schief & Rogers 1998; Liu & Mañas 1998).

4 A novel class of integrable surfaces

4.1 The geometry of O surfaces

Since the Combescure transformation preserves lines of curvature, it is possible to parametrize simultaneously any set of Combescure-related surfaces $\Sigma_\kappa \subset \mathbb{R}^3$ in terms of curvature coordinates. The orthogonality constraint

$$\vec{X} \cdot \vec{Y} = 0 \tag{14}$$

and the linear system $(9)_{1,3}$ then imply that $\vec{X} \cdot \vec{X}_y = 0$ and $\vec{Y} \cdot \vec{Y}_x = 0$. An appropriate reparametrization of the form (12) therefore results in

$$\vec{X}^2 = 1, \quad \vec{Y}^2 = 1. \tag{15}$$

Hence, it may be assumed without loss of generality that $\vec{X}$ and $\vec{Y}$ constitute orthogonal unit vectors. However, the coordinate lines on the associated dual

surfaces $\Sigma^k \subset \mathbb{R}^n$ are not necessarily orthogonal. In fact, the orthogonality condition

$$\underline{H} \cdot \underline{K} = 0 \tag{16}$$

imposes severe constraints on the surfaces Σ_κ. Thus, in this manner, one isolates a privileged class of surfaces in $\mathbb{R}^3$. It is this class of surfaces that will be the subject of the remainder of the present paper.

Definition 1 (O surfaces) *Combescure-related parametrized surfaces $\Sigma_\kappa \subset \mathbb{R}^3$ and their duals $\Sigma^k \subset \mathbb{R}^n$ are termed* (dual) O surfaces *if the coordinates on both Σ_κ and Σ^k are orthogonal.*

The above terminology is borrowed from Eisenhart (1962) who defines *O nets* as orthogonal conjugate nets. The adjoint system $(9)_{2,4}$ for dual O surfaces implies that $\underline{H}_y \cdot \underline{H} = 0$ and $\underline{K}_x \cdot \underline{K} = 0$. Once again, a suitable reparametrization of the form (13) yields

$$\underline{H}^2 = \pm 1, 0, \quad \underline{K}^2 = \pm 1, 0 \tag{17}$$

so that the assumption that $\underline{H}$ and $\underline{K}$ constitute orthogonal unit or null vectors is admissible.

Before we establish the integrability of O surfaces in the sense of the existence of a parameter-dependent linear representation and an associated Bäcklund transformation, we demonstrate below how classical surfaces such as isothermic, constant mean curvature, minimal, 'linear' Weingarten, Guichard and Petot surfaces and surfaces of constant Gaußian curvature may be retrieved as canonical examples of O surfaces.

4.2 Examples

In view of the following, it is recalled that if a surface $\Sigma \subset \mathbb{R}^3$ is parametrized in terms of curvature coordinates then the associated *principal curvatures* (Eisenhart 1960) h and k read

$$\begin{aligned} \mathsf{h} &= -\frac{\vec{R}_x \cdot \vec{N}_x}{\vec{R}_x^2} = -\frac{H_\circ}{H} \\ \mathsf{k} &= -\frac{\vec{R}_y \cdot \vec{N}_y}{\vec{R}_y^2} = -\frac{K_\circ}{K}, \end{aligned} \tag{18}$$

where $H_\circ$ and $K_\circ$ are defined by (8). In particular, the *Gaußian* and *mean curvatures* K and M respectively of the surface Σ are given by

$$\begin{aligned} \mathsf{K} &= \mathsf{hk} = \frac{H_\circ K_\circ}{HK} \\ \mathsf{M} &= \mathsf{h} + \mathsf{k} = -\frac{H_\circ}{H} - \frac{K_\circ}{K}. \end{aligned} \tag{19}$$

4.2.1 Surfaces of constant Gaußian curvature

We first consider the simplest choice

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

which corresponds to a two-dimensional Eulidean dual space $\mathbb{R}^2$. In this case, the orthogonality condition (16) takes the form

$$H_1K_1 + H_2K_2 = 0. \tag{21}$$

By virtue of $(19)_1$, this is equivalent to the requirement that the Gaußian curvatures of Σ_1 and Σ_2 be related by

$$\mathsf{K}_1 = -\mathsf{K}_2. \tag{22}$$

Alternatively, if we consider a pseudo-Euclidean dual space $\mathbb{R}^2$ with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

then the orthogonality condition reads

$$H_1K_1 = H_2K_2 \tag{24}$$

so that

$$\mathsf{K}_1 = \mathsf{K}_2. \tag{25}$$

We therefore conclude that a pair of Combescure-related surfaces parametrized in terms of curvature coordinates constitute O surfaces if, at corresponding points, their Gaußian curvatures are of the same magnitude. In particular, if we restrict the surface Σ_2 to the sphere with $\mathsf{K}_2 = 1$ then the surface Σ_1 is of constant Gaußian curvature and Σ_2 is but its spherical representation. Thus, classical *(pseudo)spherical surfaces* (Eisenhart 1960) are retrieved.

4.2.2 Isothermic and minimal surfaces

The choice

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{26}$$

leads to the orthogonality condition

$$H_1K_2 + H_2K_1 = 0. \tag{27}$$

The adjoint system $(9)_{2,4}$ then implies that $(H_1H_2)_y = 0$ and $(K_1K_2)_x = 0$. Without loss of generality, we may set

$$H_1H_2 = 1, \quad K_1K_2 = -1 \tag{28}$$

so that

$$K_1 = H_1, \quad K_2 = -H_2. \tag{29}$$

In terms of the position vectors $\vec{R}_1$ and $\vec{R}_2$ to the surfaces Σ_1 and Σ_2, the relations (28), (29) translate into

$$\vec{R}_{2x} = \frac{\vec{R}_{1x}}{\vec{R}_{1x}^2}, \quad \vec{R}_{2y} = -\frac{\vec{R}_{1y}}{\vec{R}_{1y}^2} \tag{30}$$

and the *conformality* condition

$$\vec{R}_{1x}^2 = \vec{R}_{1y}^2, \quad \vec{R}_{2x}^2 = \vec{R}_{2y}^2. \tag{31}$$

Thus, the surfaces Σ_1 and Σ_2 constitute classical *isothermic surfaces* (Eisenhart 1960) which are related by the *Christoffel transformation* (Eisenhart 1962) (30). Furthermore, if we identify Σ_2 with the spherical representation of Σ_1 then $H_2 = H_\circ$ and $K_2 = K_\circ$ which, in turn, implies that the surface Σ_1 is *minimal* (Eisenhart 1960) since

$$\mathsf{M}_1 = -\frac{H_2}{H_1} - \frac{K_2}{K_1} = 0. \tag{32}$$

The latter is in agreement with the well-known fact that the Christoffel transform of a minimal surface constitutes a sphere. It is also noted that isothermic surfaces in spaces of arbitrary dimension (Schief 2000a) may be retrieved by considering O surfaces in $\mathbb{R}^m$.

4.2.3 Constant mean curvature surfaces and the Bonnet theorem

In the preceding, we have shown how classical minimal surfaces are obtained within the framework of O surfaces. The important class of *constant mean curvature surfaces* (Eisenhart 1960), which constitute particular isothermic surfaces, may also be retrieved in a natural manner. Thus, we observe that any set of Combescure-related O surfaces Σ_κ gives rise to an infinite number of Combescure-related O surfaces by taking linear combinations of the associated position vectors $\vec{R}_\kappa$. For instance, if Σ_1 and Σ_2 are two isothermic surfaces related by the Christoffel transformation then the surfaces $\Sigma_\pm$ with position vectors

$$\vec{R}_\pm = \frac{1}{2}(\vec{R}_2 \pm \vec{R}_1) \tag{33}$$

constitute O surfaces which are Combescure transforms of both Σ_1 and Σ_2. The corresponding solutions of the adjoint system $(9)_{2,4}$ are given by

$$H_\pm = \frac{1}{2}(H_2 \pm H_1), \quad K_\pm = \frac{1}{2}(K_2 \pm K_1). \tag{34}$$

Accordingly, the Gaußian curvatures of the surfaces $\Sigma_\pm$ take the form

$$\mathsf{K}_\pm = \frac{H_\circ K_\circ}{H_\pm K_\pm} = \frac{4H_\circ K_\circ}{H_1K_1 + H_2K_2} \tag{35}$$

by virtue of (27) and hence coincide. This is not surprising since the transition from (Σ_1, Σ_2) to (Σ_+, Σ_-) may be interpreted at the level of the matrix S as a similarity transformation mapping the case (26) to the case (23).

If we now identify the surface Σ_- with the spherical representation of the isothermic surfaces, that is

$$\vec{R}_- = \vec{N}, \quad H_- = H_\circ, \quad K_- = K_\circ, \tag{36}$$

then

$$\mathsf{K}_\pm = 1, \quad \mathsf{M}_1 = 1, \quad \mathsf{M}_2 = -1. \tag{37}$$

The latter relations encapsulate a well-known theorem due to Bonnet (Eisenhart 1960) which states that with any surface Σ_+ of constant Gaußian curvature $\mathsf{K}_+ = 1$ one may associate two parallel surfaces Σ_1 and Σ_2 of constant mean curvature $\mathsf{M}_1 = 1$ and $\mathsf{M}_2 = -1$ respectively with position vectors

$$\vec{R}_1 = \vec{R}_+ - \vec{N}, \quad \vec{R}_2 = \vec{R}_+ + \vec{N}. \tag{38}$$

This may be regarded as a special case of the following statement:

If the Gaußian curvatures of two Combescure-related surfaces $\Sigma_\pm$ parametrized in terms of curvature coordinates are equal at corresponding points then the Combescure transforms Σ_1 and Σ_2 defined by

$$\vec{R}_1 = \vec{R}_+ - \vec{R}_-, \quad \vec{R}_2 = \vec{R}_+ + \vec{R}_- \tag{39}$$

constitute isothermic surfaces which are related by the Christoffel transformation.

4.2.4 ‘Linear’ Weingarten surfaces

Surfaces of constant Gaußian or mean curvature represent particular examples of *Weingarten surfaces* (Eisenhart 1960), that is surfaces in $\mathbb{R}^3$ which admit a functional relation between the principal curvatures. ‘Linear’ Weingarten surfaces are those corresponding to a functional relation of the form

$$\alpha\mathsf{K} + \beta\mathsf{M} = \gamma, \tag{40}$$

where α, β and γ are arbitrary constants. If Σ constitutes a linear Weingarten surface parametrized in terms of curvature coordinates and $\Sigma_\circ$ denotes its spherical representation then, on use of the expressions (19) for the Gaußian and mean curvatures K and M, the above relation may be brought into the form

$$\underline{H} \cdot \underline{K} = 0, \quad S = \begin{pmatrix} \gamma & \beta \\ \beta & -\alpha \end{pmatrix} \tag{41}$$

with the identification

$$\underline{H} = (H, H_\circ), \quad \underline{K} = (K, K_\circ). \tag{42}$$

Thus, linear Weingarten surfaces constitute O surfaces which are parallel to surfaces of constant Gaußian curvature since the matrix S as given by $(41)_2$ may be mapped by means of an appropriate similarity transformation to either (20) or (23) provided that $\det S \neq 0$. At the level of the position matrix $\underline{\vec{R}}$, this corresponds to a linear transformation.

4.2.5 Guichard surfaces

If we equip a three-dimensional dual space with the indefinite metric

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{43}$$

then the orthogonality condition becomes

$$H_1 K_2 + H_2 K_1 + H_3 K_3 = 0. \tag{44}$$

The corresponding O surfaces in $\mathbb{R}^3$ evidently generalize isothermic surfaces. If Σ_3 is taken to be the spherical representation of Σ_1 and Σ_2 then the relations $H_3 = H_\circ$ and $K_3 = K_\circ$ imply that

$$\frac{1}{\mathsf{h}_1 \mathsf{k}_2} + \frac{1}{\mathsf{h}_2 \mathsf{k}_1} + 1 = 0. \tag{45}$$

Accordingly, Σ_1 and Σ_2 represent *Guichard surfaces* (Guichard 1900; Eisenhart 1962) as alluded to in the introduction.

4.2.6 Petot surfaces

Another canonical class of O surfaces is obtained by identifying the three-dimensional Euclidean space with its dual. Thus, if we set

$$\underline{H} = \vec{X}^{\intercal}, \quad \underline{K} = \vec{Y}^{\intercal}, \quad p = q \tag{46}$$

then the linear systems (9) coincide. The constraint $(46)_3$ is known to define *Petot surfaces* (Petot 1891). Accordingly, the O surfaces Σ_κ constitute three Petot surfaces which are linked by Combescure transformations. Moreover, the defining relations for the 'position matrix' $\underline{\vec{R}}$ read

$$\underline{\vec{R}}_x = \vec{X}\vec{X}^{\intercal}, \quad \underline{\vec{R}}_y = \vec{Y}\vec{Y}^{\intercal} \tag{47}$$

which, in turn, imply the 'conservation laws'

$$(\vec{R}^2_{\kappa x})_y = (\vec{R}^2_{\kappa y})_x. \tag{48}$$

The metrics on the surfaces Σ_κ may therefore be derived from potentials, that is

$$ds^2_\kappa = d\vec{R}_\kappa \cdot d\vec{R}_\kappa = \varphi_{\kappa x} dx^2 + \varphi_{\kappa y} dy^2. \tag{49}$$

The latter property constitutes an alternative characterization of Petot surfaces. In fact, it reflects the fact that Petot surfaces represent the constituent members of Darboux-Egorov-type triply orthogonal systems of surfaces (Egorov 1900, 1901). It is noted, however, that the particular form of the potentials φ_κ, namely[3]

$$\varphi_\kappa = R_\kappa^\kappa, \tag{50}$$

indicates that the above Petot surfaces are not generic. The class of Petot O surfaces nevertheless enshrines the generic class of Petot surfaces in the sense that any Petot surface may be obtained from a Petot O surface by application of an appropriate Combescure transformation.

5 A Bäcklund transformation for O surfaces

An extensive account of the transformation theory of conjugate nets is contained in the treatise 'Transformations of Surfaces' by Eisenhart (1962). Here, we focus on the classical *Fundamental transformation* (Jonas 1915, Eisenhart 1962). Since the Fundamental transformation commutes with the Combescure transformation, it can be simultaneously applied to sets of Combescure-related surfaces.

5.1 The Fundamental and Ribaucour transformations

The Fundamental transformation is generated by two pairs of scalar solutions of the linear systems (9) and corresponding bilinear potentials of the form (10). Thus, for a given pair of functions p, q associated with a set of Combescure-related surfaces Σ_κ, let $\{X, Y\}$ and $\{H, K\}$ be solutions of the linear systems

$$\begin{aligned} X_y &= qY, \quad & H_y &= pK \\ Y_x &= pX, \quad & K_x &= qH. \end{aligned} \tag{51}$$

In the sequel, we refer to X, Y and H, K as *eigenfunctions* and *adjoint eigenfunctions* respectively. Three bilinear potentials $\vec{M}, \underline{M}$ and M may now be introduced according to

$$\begin{aligned} \vec{M}_x &= \vec{X}H, \quad & \underline{M}_x &= X\underline{H}, \quad & M_x &= XH \\ \vec{M}_y &= \vec{Y}K, \quad & \underline{M}_y &= Y\underline{K}, \quad & M_y &= YK. \end{aligned} \tag{52}$$

A second set of Combescure-related surfaces Σ'_κ is now obtained as follows:

Theorem 1 (The Fundamental transformation) *The linear systems (9) and the defining relations (10) are invariant under*

$$(\underline{\vec{R}}, \vec{X}, \vec{Y}, \underline{H}, \underline{K}, p, q) \to (\underline{\vec{R}}', \vec{X}', \vec{Y}', \underline{H}', \underline{K}', p', q'), \tag{53}$$

[3]For simplicity, we here use the normalization (15).

where

$$\underline{\vec{R}}' = \underline{\vec{R}} - \frac{\vec{M}\underline{M}}{M} \tag{54}$$

and

$$\begin{aligned} \vec{X}' &= \vec{X} - \frac{X\vec{M}}{M}, \quad & \vec{Y}' &= \vec{Y} - \frac{Y\vec{M}}{M} \\ \underline{H}' &= \underline{H} - \frac{H\underline{M}}{M}, \quad & \underline{K}' &= \underline{K} - \frac{K\underline{M}}{M} \\ p' &= p - \frac{YH}{M}, \quad & q' &= q - \frac{XK}{M}. \end{aligned} \tag{55}$$

It is emphasized that the above transformation may also be regarded as a mapping between the sets of dual surfaces Σ^k and Σ'^k.

If the coordinate lines on Σ_κ are lines of curvature, that is $\vec{X} \cdot \vec{Y} = 0$, then it is readily verified that the quantities

$$X = \vec{M} \cdot \vec{X}, \quad Y = \vec{M} \cdot \vec{Y} \tag{56}$$

constitute particular eigenfunctions. This choice of eigenfunctions in the definitions of the bilinear potentials $\vec{M}$ and M leads, in turn, to the relations

$$(\vec{M}^2)_x = 2M_x, \quad (\vec{M}^2)_y = 2M_y \tag{57}$$

so that we may set

$$\vec{M}^2 = 2M. \tag{58}$$

It is now straightforward to show that

$$\vec{X}'^2 = \vec{X}^2, \quad \vec{X}' \cdot \vec{Y}' = 0, \quad \vec{Y}'^2 = \vec{Y}^2. \tag{59}$$

Thus, it turns out that lines of curvature and the normalisation (15) are preserved by the Fundamental transformation if the eigenfunctions X, Y and the bilinear potential M are chosen to be (56) and (58) respectively. Under these circumstances, the Fundamental transformation becomes the classical *Ribaucour transformation* (Eisenhart 1962).

5.2 Application to O surfaces

It is remarkable that the Ribaucour transformation may be constrained in such a way that orthogonality of the coordinate lines on the dual surfaces is also sustained. In fact, as a by-product, a parameter-dependent linear representation of O surfaces is obtained. As in the preceding, we first observe that the quantities

$$H = \lambda \underline{M} \cdot \underline{H}, \quad K = \lambda \underline{M} \cdot \underline{K} \tag{60}$$

constitute particular adjoint eigenfunctions. The constant parameter λ is now non-trivial as we have already specified the eigenfunctions X and Y. The associated potentials $\underline{M}$ and M then obey the relations

$$\lambda(\underline{M}^2)_x = 2M_x, \quad \lambda(\underline{M}^2)_y = 2M_y \tag{61}$$

so that

$$\lambda\underline{M}^2 = 2M \tag{62}$$

is, at least, consistent. It is shown below that this constraint is indeed admissible. On this assumption, we now proceed and note that

$$\underline{H}'^2 = \underline{H}^2, \quad \underline{H}' \cdot \underline{K}' = 0, \quad \underline{K}'^2 = \underline{K}^2, \tag{63}$$

which implies that the coordinate lines are also orthogonal on the transformed dual surfaces Σ'^k with the normalisation (17) unchanged.

Finally, insertion of the (adjoint) eigenfunctions X, Y and H, K as given by (56) and (60) respectively into the defining relations (52) produces the following *Lax pair* for O surfaces:

Theorem 2 (A Lax pair for O surfaces) *The linear system*

$$\begin{aligned}
\begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix}_x &= \begin{pmatrix} 0 & \lambda\vec{X}\underline{H} \\ \underline{H}^{\intercal}\vec{X}^{\intercal} & 0 \end{pmatrix} \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix} \\
\begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix}_y &= \begin{pmatrix} 0 & \lambda\vec{Y}\underline{K} \\ \underline{K}^{\intercal}\vec{Y}^{\intercal} & 0 \end{pmatrix} \begin{pmatrix} \vec{M} \\ \underline{M}^{\intercal} \end{pmatrix}
\end{aligned} \tag{64}$$

is compatible modulo the linear systems (9) and the orthogonality conditions (14) and (16). It admits the first integral

$$\vec{M}^2 - \lambda\underline{M}^2 = \text{const.} \tag{65}$$

The existence of the first integral (65) guarantees that the constraint (62) is admissible. Consequently, we are now in a position to formulate the following theorem:

Theorem 3 (A Bäcklund transformation for O surfaces) *Let $\underline{\vec{R}}$ be the position matrix of a set of Combescure-related O surfaces Σ_κ and their duals Σ^k and $\vec{X}, \vec{Y}, \underline{H}, \underline{K}$ corresponding tangent vectors. If the vectors $\vec{M}$ and $\underline{M}$ constitute a solution of the linear system (64) subject to the admissible constraint*

$$\vec{M}^2 = \lambda\underline{M}^2 = 2M \tag{66}$$

and the scalar M is defined by the latter then the position matrix of a second set of O surfaces $\Sigma'_\kappa, \Sigma'^k$ is given by

$$\underline{\vec{R}}' = \underline{\vec{R}} - \frac{\vec{M}\underline{M}}{M}. \tag{67}$$

We conclude this section with two remarks. Firstly, if the O surface Σ_n is identified with the spherical representation of the remaining O surfaces Σ_κ then

$$\vec{R}_n^2 = 1. \tag{68}$$

In this case, by differentiation, it is readily verified that

$$M_n = \vec{R}_n \cdot \vec{M} \tag{69}$$

is another admissible constraint. Consequently, the nth component of the transformation law (67) may be cast into the form

$$\vec{R}'_n = \left(\mathbb{1} - 2\frac{\vec{M}\vec{M}^{\intercal}}{\vec{M}^2}\right)\vec{R}_n \tag{70}$$

which implies that

$$\vec{R}'^2_n = 1. \tag{71}$$

Hence, we come to the important conclusion that the above Bäcklund transformation acts within specific sub-classes of O surfaces such as (pseudo)spherical, minimal or Guichard surfaces. Moreover, it is readily shown that constraints of the form

$$\left(\sum_{\kappa=1}^{n} c_\kappa \vec{R}_\kappa\right)^2 = 1, \tag{72}$$

which generalize (68), may also be preserved. In particular, the specialization (36) leading to constant mean curvature surfaces proves invariant.

Secondly, in Hertrich-Jeromin & Pedit (1997), it has been pointed out that the classical Bäcklund transformation for isothermic sufaces may be formulated in terms of a *matrix Riccati system*. It turns out that such a system is generic. Thus, differentiation of (67) and use of (10) lead to a matrix Riccati system for the new position matrix $\underline{\vec{R}}'$, namely

$$(\Delta\underline{\vec{R}})_{x^i} = -\frac{\lambda}{2}(\underline{\vec{R}}_{x^i}\Delta\underline{\vec{R}}^{\intercal}\Delta\underline{\vec{R}} + \Delta\underline{\vec{R}}\Delta\underline{\vec{R}}^{\intercal}\underline{\vec{R}}_{x^i}) + \lambda\Delta\underline{\vec{R}}\,\mathrm{tr}\,(\underline{\vec{R}}_{x^i}\Delta\underline{\vec{R}}^{\intercal}), \tag{73}$$

where $\Delta\underline{\vec{R}} = \underline{\vec{R}}' - \underline{\vec{R}}$ and $(x^1, x^2) = (x, y)$. It is emphasized that, remarkably, the above first-order system only involves the seed position matrix, its Bäcklund transform and an arbitrary constant Bäcklund parameter. If one sets aside the genesis of the pair (73), it would be interesting to determine whether its compatibilty conditions imply that $\underline{\vec{R}}'$ and $\underline{\vec{R}}$ necessarily define O surfaces. In the terminology of integrable systems, a pair of this type is termed a *strong* Bäcklund transformation.

6 The Gauß-Mainardi-Codazzi equations

In the preceding, contact has been made with classical isothermic surfaces in the context of particular O surfaces and a Riccati-type formulation of the Bäcklund transformation for O surfaces. It turns out that this connection may be exploited to obtain an alternative linear representation for O surfaces which generalizes that for isothermic surfaces obtained by Darboux (1899) (Eisenhart 1962). We

first note that the *Gauß-Weingarten equations* (Eisenhart 1960) for surfaces in $\mathbb{R}^3$ parametrized in terms of lines of curvature are encoded in the linear system

$$\begin{aligned}
\begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{N} \end{pmatrix}_x &= \begin{pmatrix} 0 & -p & -H_\circ \\ p & 0 & 0 \\ H_\circ & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{N} \end{pmatrix} \\
\begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{N} \end{pmatrix}_y &= \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -K_\circ \\ 0 & K_\circ & 0 \end{pmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{N} \end{pmatrix}
\end{aligned} \tag{74}$$

due to the orthogonality condition $\vec{X} \cdot \vec{Y} = 0$. The compatibility condition for this system produces the *Gauß-Mainardi-Codazzi equations*

$$p_y + q_x + H_\circ K_\circ = 0, \quad H_{\circ y} = pK_\circ, \quad K_{\circ x} = qH_\circ. \tag{75}$$

If the surfaces constitute Combescure-related O surfaces Σ_κ then the underdetermined system (75) is coupled with the relations

$$\underline{H} \cdot \underline{K} = 0, \quad \underline{H}_y = p\underline{K}, \quad \underline{K}_x = q\underline{H} \tag{76}$$

which completes the set of Gauß-Mainardi-Codazzi equations. It is now directly verified that the system (75), (76) holds if and only if the linear system

$$\begin{aligned}
\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{N} \\ \underline{\mathcal{R}}^\top \end{pmatrix}_x &= \begin{pmatrix} 0 & -p & -H_\circ & m\underline{H} \\ p & 0 & 0 & 0 \\ H_\circ & 0 & 0 & 0 \\ \underline{H}^\top & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{N} \\ \underline{\mathcal{R}}^\top \end{pmatrix} \\
\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{N} \\ \underline{\mathcal{R}}^\top \end{pmatrix}_y &= \begin{pmatrix} 0 & q & 0 & 0 \\ -q & 0 & -K_\circ & m\underline{K} \\ 0 & K_\circ & 0 & 0 \\ 0 & \underline{K}^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{N} \\ \underline{\mathcal{R}}^\top \end{pmatrix}
\end{aligned} \tag{77}$$

is compatible. Here, m is an arbitrary constant parameter. Moreover, for $m = 0$, the Gauß-Weingarten equations (74) together with the defining relations (10) for the position matrix of O surfaces are retrieved if one considers vector-valued solutions of the linear system (77). It is also noted that the above parameter-dependent linear representation for O surfaces indeed reduces to that for isothermic surfaces in the case (26).

As an illustration, we focus on the case (20), that is

$$\underline{H}^2 = 1, \quad \underline{K}^2 = 1, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{78}$$

A convenient parametrization of the tangent vectors $\underline{H}$ and $\underline{K}$ is then given by

$$H_1 = -K_2 = \cos\theta, \quad K_1 = H_2 = \sin\theta. \tag{79}$$

Accordingly, the system (76) reduces to

$$p = -\theta_y, \quad q = \theta_x \tag{80}$$

and the Gauß-Mainardi-Codazzi equations become

$$\theta_{xx} - \theta_{yy} + H_\circ K_\circ = 0, \quad H_{\circ y} = -\theta_y K_\circ, \quad K_{\circ x} = \theta_x H_\circ. \tag{81}$$

If we now identify the surface Σ_2 with the spherical representation of Σ_1 then $H_\circ = H_2$, $K_\circ = K_2$ and the classical sine-Gordon equation

$$\theta_{xx} - \theta_{yy} = \sin\theta\cos\theta \tag{82}$$

underlying pseudospherical surfaces is recovered (Eisenhart 1960).

7 The pseudosphere and breather pseudospherical surfaces

We conclude this paper with an illustration of the Bäcklund transformation for O surfaces and consider the particular case of pseudospherical surfaces as discussed in the previous section. Thus, we here regard a straight line as a (degenerate) seed pseudospherical surface Σ_1 together with its 'spherical representation' Σ_2 represented by

$$\vec{R}_1 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, \quad \vec{R}_2 = \begin{pmatrix} -\sin y \\ \cos y \\ 0 \end{pmatrix} \tag{83}$$

so that the tangent vectors to Σ_1, Σ_2 and their duals read

$$\vec{X} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} \cos y \\ \sin y \\ 0 \end{pmatrix}, \quad \underline{H} = (1 \;\; 0), \quad \underline{K} = (0 \;\; -1). \tag{84}$$

It is evident that the linear systems (9) with $p = q = 0$ and the orthogonality conditions $\vec{X}\cdot\vec{Y} = \underline{H}\cdot\underline{K} = 0$ are satisfied. Accordingly, the linear system (64) for these particular O surfaces becomes

$$\begin{aligned} \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}_x &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix} \\ \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}_y &= \begin{pmatrix} 0 & 0 & 0 & 0 & -\lambda\cos y \\ 0 & 0 & 0 & 0 & -\lambda\sin y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\cos y & -\sin y & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M^1 \\ M^2 \\ M^3 \\ M_1 \\ M_2 \end{pmatrix}. \end{aligned} \tag{85}$$

The latter decouples into two systems of linear ordinary differential equations for $M^3(x), M_1(x)$ and $M^1(y), M^2(y), M_2(y)$ respectively. The constants of integration in the general solution of (85) have to be chosen in such a way that the admissible constraints (66) and $(69)_{n=2}$ are satisfied. For brevity, we here only state the result of this analysis:

In the case $\lambda = 1$, the position vector $\vec{R}_1'$ of the Bäcklund transform Σ_1' may be reduced to

$$\vec{R}_1' = \begin{pmatrix} \dfrac{\sin y}{\cosh x} \\ -\dfrac{\cos y}{\cosh x} \\ x - \tanh x \end{pmatrix}. \tag{86}$$

This pseudsospherical surface of revolution is nothing but Beltrami's classical *pseudosphere* (Eisenhart 1960) as depicted in Figure 1.

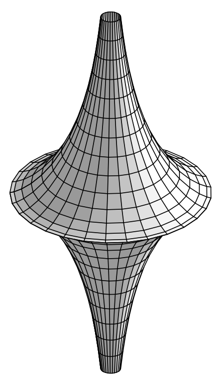

Figure 1: The classical pseudosphere

If $\lambda \neq 1$ then integration of the linear system (85) and specification of the constants of integration lead to the position vector

$$\vec{R}_1' = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} - \frac{2d}{c} \frac{\cosh cx}{c^2 \sin^2 dy + d^2 \cosh^2 cx} \begin{pmatrix} -\sin dy \sin y - d \cos dy \cos y \\ \sin dy \cos y - d \cos dy \sin y \\ d \sinh cx \end{pmatrix}, \tag{87}$$

where

$$\lambda = c^2, \quad c^2 + d^2 = 1. \tag{88}$$

These pseudospherical surfaces are associated with the 'stationary' *breather* solutions of the sine-Gordon equation (82) if the constants c and d are real. There exists a discrete rotational symmetry if d is rational, that is

$$d = \frac{\mathsf{p}}{\mathsf{q}}, \quad \mathsf{p}, \mathsf{q} \in \mathbb{Z}. \tag{89}$$

A variety of *breather pseudospherical surfaces* (Rogers & Schief 2000) corresponding to different choices of p and q is displayed in Figure 2.

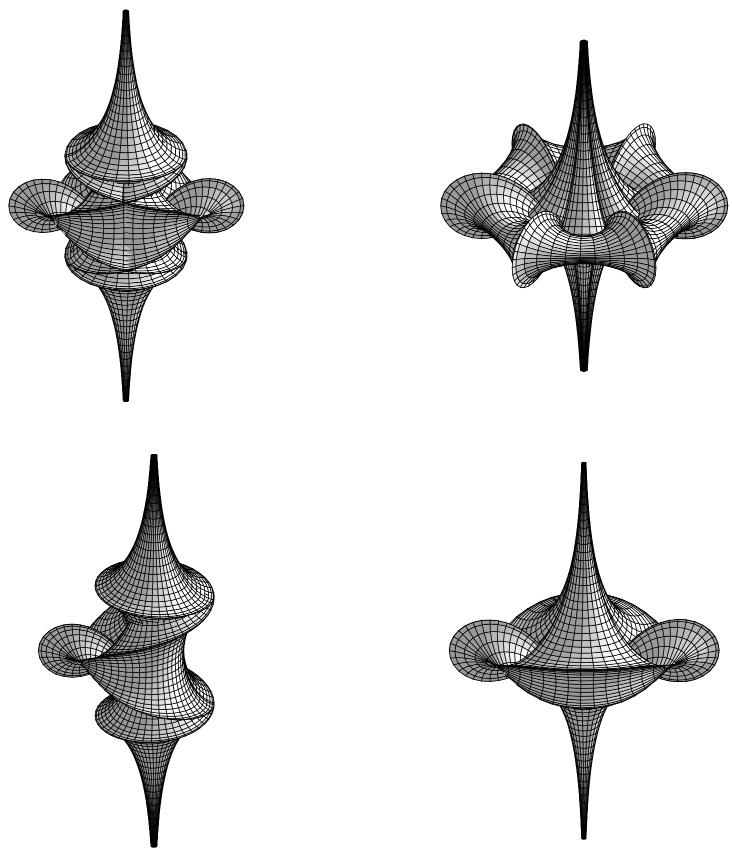

Figure 2: Breather pseudospherical surfaces: $\frac{\mathsf{p}}{\mathsf{q}} = \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{1}{2}$

It is interesting to note that the Bäcklund transformation for O surfaces does not reduce to the *classical* Bäcklund transformation for pseudospherical surfaces. In fact, as discussed above, a single application of the Bäcklund transformation for O surfaces to a straight line produces the pseudosphere or breather pseudospherical surfaces. Here, it is required to solve a system of linear *non-autonomous* differential equations. By contrast, a single application of the classical Bäcklund transformation involves the solution of a *constant-coefficient* linear system and results in a one-parameter family of *Dini surfaces* including the pseudosphere (Eisenhart 1960). A second application, which is purely algebraic in nature due to the existence of an associated *permutability theorem* (Eisenhart 1960), then leads to breather pseudospherical surfaces if one assumes that the two Bäcklund parameters are complex conjugates (Rogers & Schief 2000).

8 Concluding remark

It is evident that any position matrix $\vec{\underline{R}}$ associated with O surfaces may be interpreted as a $3n$-dimensional vector composed of the entries R^k_κ. Thus, $\vec{\underline{R}}$ defines a surface Σ in a $3n$-dimensional pseudo-Euclidean space endowed with a metric determined by the symmetric matrix S. If, for simplicity, we consider the Euclidean case of a unit matrix S with

$$\underline{H}^2 = 1, \quad \underline{K}^2 = 1 \tag{90}$$

then the induced metric on Σ is 'flat', that is

$$\mathrm{I} = < d\vec{\underline{R}}, d\vec{\underline{R}} > = dx^2 + dy^2. \tag{91}$$

Canonical second fundamental forms are obtained by choosing an orthonormal basis $\underline{N}^\alpha$, $\alpha = 3, \ldots n$ of the normal bundle associated with the dual O surfaces Σ_κ satisfying

$$\underline{N}^\alpha_x = X^\alpha \underline{H}, \quad \underline{N}^\alpha_y = Y^\alpha \underline{K} \tag{92}$$

with some functions X^α and Y^α. Since

$$\underline{H} \cdot \underline{N}^\alpha = 0, \quad \underline{K} \cdot \underline{N}^\alpha = 0, \quad \underline{N}^\alpha \cdot \underline{N}^\beta = \delta^{\alpha\beta}, \tag{93}$$

where $\delta^{\alpha\beta}$ denotes the usual Kronecker symbol, the quantities

$$\begin{gathered} \vec{\underline{N}}^2_1 = \vec{X}\underline{K}, \quad \vec{\underline{N}}^1_2 = \vec{Y}\underline{H}, \quad \vec{\underline{N}}^1_3 = \vec{N}\underline{H}, \quad \vec{\underline{N}}^2_3 = \vec{N}\underline{K} \\ \vec{\underline{N}}^\alpha_1 = \vec{X}\underline{N}^\alpha, \quad \vec{\underline{N}}^\alpha_2 = \vec{Y}\underline{N}^\alpha, \quad \vec{\underline{N}}^\alpha_3 = \vec{N}\underline{N}^\alpha \end{gathered} \tag{94}$$

are readily shown to form an orthonormal basis of the $(3n-2)$–dimensional normal bundle attached to Σ. The corresponding second fundamental forms

$$\mathrm{II}^\kappa_k = - < d\vec{\underline{R}}, d\vec{\underline{N}}^\kappa_k > \tag{95}$$

therefore become

$$\begin{gathered} \mathrm{II}^2_1 = -q\,dx^2 + 2p\,dxdy - q\,dy^2, \quad \mathrm{II}^1_2 = -p\,dx^2 + 2q\,dxdy - p\,dy^2 \\ \mathrm{II}^1_3 = -H_\circ dx^2, \quad \mathrm{II}^2_3 = -K_\circ dy^2 \\ \mathrm{II}^\alpha_1 = -X^\alpha dx^2, \quad \mathrm{II}^\alpha_2 = -Y^\alpha dy^2, \quad \mathrm{II}^\alpha_3 = 0. \end{gathered} \tag{96}$$

The geometry of the surfaces $\Sigma \subset \mathbb{R}^{3n}$ defined via Combescure-related O surfaces is currently being investigated.

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